A Utility Representation Theorem for General Revealed Preference

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Abstract

We provide a utility representation theorem for the revealed preference of an agent choosing in an arbitrary space endowed with a separable partial order. The result can be applied to construct new revealed preference tests for choices over infinite consumption streams and probability distribution spaces, among other cases of interest in economics. As an illustration, we construct revealed preference tests for best-responding behavior in strategic games and infinite horizon consumption problems.

Keywords: revealed preferences, representation theorem, preference extensions, equilibrium play, intertemporal consumption

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1. Introduction

We provide a necessary and sufficient revealed preference condition for an observed set of choices to be generated by maximization of a utility function. The condition only requires the space to be endowed with a separable partial order. Our result extends the scope of applications of revealed preference tests to spaces which are not covered by existing results, including infinite consumption streams and probability distributions (including mixed strategies in games). We illustrate our approach by constructing a revealed preference test for best-responding behavior in strategic games and infinite horizon consumption problems.

Starting with Samuelson (1938), Richter (1966), and Afriat (1967), revealed preference literature seeks to test whether an observed set of choices can be consistent with maximization of a utility function. The central premise is that we can only observe choices and not the entire preference relation. Revealed preference theory allows data to speak for itself and therefore avoids the problem of parametric misspecification of preferences. Chambers and Echenique (2016) offer a general review of the revealed preference approach and its use for testing theories of individual behavior.

There is a growing interest in developing a comprehensive approach to revealed preference that can be applied in a wide variety of contexts of interest. There are two related strands of research. The first of them deals with preference extensions in an abstract setting. Suzumura (1976) Duggan (1999), and Demuynck (2009) provide preference extension theorems and their links with consistency conditions in terms of revealed preferences. A drawback of using a completely abstract framework is obtaining a preference relation which may not be representable by a utility function.

The second strand of literature generalizes the classical result of Afriat (1967). In this line, Forges and Minelli (2009) generalizes the applicability of the Afriat test to nonlinear budget sets. Nishimura et al. (2017) extends Afriat results to general topological spaces instead of the standard assumption of a real hyperplane. A fundamental assumption behind results in this line of research is local compactness of the topological space. This assumption does not necessarily hold for important settings like the spaces containing infinite consumption streams and probability measures.

This paper attempts to close the gap between the two strands of literature. In particular, we construct an extension of the preference relation that can be represented by a utility function under the minimal assumption of separability of the partial order associated to the space of alternatives. This assumption is similar to making the separability assumption over the natural topology of this order.[1]

The remainder of this paper is organized as follows. Section 2 contains the basic definitions. Section 3 shows the main result and its application to construct the test of best-responding behavior in static games. Section 4 provides concluding remarks. All proofs omitted in the text are collected in an Appendix.

[1]Formally, our assumption is even weaker since the topology we require to be separable can be coarser (smaller).

2. Preliminaries

Consider a partially ordered space[2] $(X, \geq)$ representing the universal set of alternatives. Denote by $> \subseteq \geq$ the **strict part** of $\geq$, that is the part that is asymmetric[3] and transitive. A partial order $\geq$ is said to be **separable** if there is a countable set $Z \subseteq X$ such that for every $y > x$ there is $z \in Z$ such that $y \geq z \geq x$.[4] A utility function $u : X \to \mathbb{R}$ is said to be **monotonic** if $x \geq y$ implies $u(x) \geq u(y)$ and $x > y$ implies $u(x) > u(y)$.

Let $\mathcal{B}$ be a collection of subsets over X, representing possible budget sets. Denote by $C : \mathcal{B} \to 2^X$ a **choice correspondence** over $\mathcal{B}$ assigning to each $B \in \mathcal{B}$ a nonempty set $C(B) \subseteq B$. Further we refer to $C(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} C(B)$ as to the collection of all chosen points. A **data set** is a tuple $(\mathcal{B}, C)$ that assigns choices to every budget from a collection. A data set $(\mathcal{B}, C)$ is **rationalizable** if there is a monotonic utility function $u : X \to \mathbb{R}$ such that $u(x) \geq u(y)$ for every $y \in B$ and $x \in C(B)$ for every $B \in \mathcal{B}$.

Denote by

$$B^{\geq} = \{y \in X : \text{there is } x \in B \text{ such that } y \leq x\}$$

the downward closure of budget B, and by

$$B^{>} = \{y \in X : \text{there is } x \in B \text{ such that } y < x\}$$

the interior of the downward closure of B.

Definition 1. *A data set $(\mathcal{B}, C)$ satisfies the* **Generalized Axiom of Revealed Preferences (GARP)** *if for every sequence $x_1 \in C(B_1), \ldots, x_n \in C(B_n)$, $B_1, \ldots, B_n \in \mathcal{B}$,*

$$x_{t+1} \in B_t^{\geq} \text{ for every } t = 1, \ldots, n-1 \text{ implies } x_1 \notin B_n^{>}.$$

[2] A partially ordered space is a set X equipped with a partial order $\geq$, that is a binary relation $\geq \subseteq X \times X$ that is reflexive ($\forall x \in X\ x \geq x$), antisymmetric ($\forall x, y \in X\ x \geq y\ \&\ y \geq x \Rightarrow x = y$) and transitive ($\forall x, y, z \in X\ x \geq y\ \&\ y \geq z \Rightarrow x \geq z$).

[3] A binary relation is said to be asymmetric if $x \geq y$ implies $y \not\geq x$ for every $x, y \in X$.

[4] This condition is sometimes referred as Debreu separability in the literature (see e.g. Herden and Levin, 2012) in order to avoid confusion with the stronger condition of Cantor separability.

3. Results

3.1. A general revealed preference condition

Theorem 1. *Let X be a space of alternatives endowed with the separable partial order $\geq$. A data set $(\mathcal{B}, C)$ is rationalizable if and only if it satisfies GARP.*

The starting point for the proof of the theorem is an argument from the preference extension literature. We consider revealed preference as an incomplete preference relation, and show that there is a converging algorithm leading to a complete preference relation. Moreover, the algorithm guarantees the limit preference relation is transitive and separable, which guarantees the existence of a utility representation (see Debreu, 1954). While our definition of GARP incorporates monotonicity, the algorithm can incorporate other desiderata such as homotheticity and quasi-linearity.

3.2. Revealed best responses

Let $I = \{1, \dots, k\}$ be a set of players, let S_i be a countable set of pure strategies available to player $i \in N$, let $S = \times_{i \in I} S_i$ be the set of strategy profiles, and let $\phi_i : S \to \mathbb{R}$ be the monetary payoff function of player $i \in I$. Abusing terminology, we refer to the triple $G = \langle N, S, (\phi_i) \rangle$ as a game form, though we have specified only the monetary payoffs of the players, not their actual (unobserved) utility payoffs. From now on we assume that players are concerned exclusively with their own monetary payoffs—the more money the better.

Let $\sigma_i \in \triangle(S_i)$ be a finite support mixed strategy for player $i \in I$, and let $\sigma = \times_{i \in I} \sigma_i$ be a profile of mixed strategies. By finite support we mean that each mixed strategy only uses a finite number of pure strategies with positive probability. Each profile of strategies induces a profile of lotteries over monetary payoffs—a single lottery for each player. We denote by $\mathcal{L}_i$ the space of monetary lotteries for player i generated by profiles of mixed strategies. Since we consider only countable sets of strategies we can restrict our attention to the space of lotteries with countably many outcomes.

We can describe the lottery over monetary payoffs for player i induced by σ by its cumulative distribution function $F_{i,\sigma} : \mathbb{R} \to [0, 1]$. Note that $F_{i,\sigma}$ satisfies the usual properties, i.e., it is nondecreasing, right-continuous, and satisfies $\lim_{x \to -\infty} F_{i,\sigma}(x) = 0$ and $\lim_{x \to +\infty} F_{i,\sigma}(x) = 1$. A lottery $F_{i,\sigma}$ first-order stochastically dominates a lottery $F_{i,\sigma'}$ (denoted by $F_{i,\sigma} \geq_{FSD} F_{i,\sigma'}$)

if $F_{i,\sigma}(x) \leq F_{i,\sigma'}(x)$ for every $x \in \mathbb{R}$. Next we define the strict first-order stochastic dominance relation. Let $F_{i,\sigma} >_{FSD} F_{i,\sigma'}$ if $F_{i,\sigma}(x) < F_{i,\sigma'}(x)$ for every x such that $0 < F_{i,\sigma}(x) < 1$. Bringing back the framework to the revealed preference setup, we only need to consider unilateral deviations to investigate best-responding behavior. The budget set from which a player makes a choice is determined by the strategies of other players. Let

$$B_i = \{F_{i,\sigma_i,\sigma_{-i}} \in \mathcal{L}_i : \sigma_i \in \triangle(S_i)\}$$

given $\sigma_{-i} \in \triangle(S_{-i})$. We denote the downward closure of the budget set by

$$B_i^{\geq_{FSD}} = \{F_i \in \mathcal{L}_i : \text{ there is } F_i' \in B_i \text{ such that } F_i' \geq_{FSD} F_i\}$$

and the strict downward closure by

$$B_i^{>_{FSD}} = \{F_i \in \mathcal{L}_i : \text{ there is } F_i' \in B_i \text{ such that } F_i' >_{FSD} F_i\}.$$

Next, we define the data set. We observe a collection of mixed strategy profiles from a collection of games. Given a game G^t, the actions of other players σ_{-i}^t and the strategies available to a player $i \in I$ determine the player's budget set B_i^t.

Let us briefly explain which parts can and what cannot vary within the data set. First, we keep the universal set of strategies S_i and therefore universal strategy profiles to be constant as well as the universal mapping $\phi_i : S_i \to \mathbb{R}$ from strategies to the payoffs.[5] Hence, an instance of a game can be uniquely determined by selecting a subset of strategies for each player S_i^t for every $i \in I$. This subset would define the corresponding subset o strategy profiles and payoffs. We keep the number of players to be fixed for the simplicity of the argument, but in general even that can change.

Hence, a data set can be defined as a triple $D = \{(G^t, \sigma^t, B_i^t)\}_{t \in T}$. A data set is said to be **rationalizable with best-responding behavior for player** i if there is a monotonic utility function $U_i : \mathcal{L}_i \to \mathbb{R}$ such that $U_i(F_{i,\sigma_i^t,\sigma_{-i}^t}) \geq U_i(F_{i,\sigma_i',\sigma_{-i}^t})$ for every $\sigma_i' \in \triangle(S_i^t)$ for every observation $t \in T$. Note that we do not require expected utility maximization.

[5]There is no restriction coming with this assumption, because if one wants to add a different payoff to the strategy profile then it is enough to add the instance of strategy to the universal set.

Corollary 1. *The data set* $D = \{(G^t, \sigma^t, B_i^t)\}_{t \in T}$ *is rationalizable with best-responding behavior for player* i *if and only if for every sequence* $1, \ldots, n \in T$ *such that*

$$F_{i,\sigma^{t+1}} \in [B_i^t]^{\geq_{FSD}} \text{ for every } t \in \{1, \ldots, n-1\} \text{ implies } F_{i,\sigma^1} \notin [B_i^n]^{>_{FSD}}.$$

The condition in Corollary 1 adapts GARP to this framework.[6] Moreover, this condition is necessary as long as we observe only an approximation of the mixed strategies. If we could observe the actual mixed strategies adopted by every player, then testing best-responding behavior for every player $i \in I$ would result in a test of equilibrium play.

We are not the first to present revealed preference conditions for games. The literature can be roughly split into two strands. The first strand starts with Sprumont (2000), who assumes that observations go beyond just the group choice from the game. In particular, Sprumont requires observing the projections of the game. This idea has been developed further by Lee (2012), who provides revealed preference conditions for equilibrium play in zero-sum games and Ray and Zhou (2001) and Ray and Snyder (2013), who provide conditions for Nash and subgame perfect equilibrium rationalization in dynamic games. The second strand in the literature assumes that we only observe group choices, but concentrates in particular classes of games. In this line, Carvajal et al. (2013) characterize revealed preference implications for Cournot competition, Chambers and Echenique (2014) develop tests of consistency with different bargaining theories, Agatsuma (2016) provides conditions for the core in transferable utility (market) games, and Cherchye et al. (2017a) and Cherchye et al. (2017b) provide criteria for the stability of marriage market.

In a sense, we combine both approaches. We deal with a collection of arbitrary games and provide conditions for mixed strategy best-responding in the collection of games. Hence, on the one hand, we operate at a level of generality which is closer to the first strand. On the other hand, as in the second strand, we require to observe only group choices albeit in different games.

[6] GARP in this case implies that choice is undominated ($F_{i,\sigma^t} \notin [B_i^t]^{>_{FSD}}$), which follows from taking the trivial sequence t_1, t_2 such that $t_1 = t_2$.

3.3. Intertemporal consumption

Denote by $T \subseteq \mathbb{N}$ the time horizon along which the agent makes her decisions. Denote by $X \subseteq \mathbb{R}^{|T|}$ the vector of intertemporal consumption streams. We assume that for every $x \in X$ there is a time period after which the agent "retires", that is the her consumption stream stabilizes. That is, for every $x \in T$ there is t_r such that $x_t = x_{t_r}$ for every $t \geq t_r$. We define a partial order as $x \geq x'$ if $x_t \geq x'_t$ for every $t \in T$. The strict part of the partial relation is defined as $x > x'$ if $x_t > x'_t$ for every $t \in T$. Note that in general the time of "retirement" may differ between different observations.

Given the defined space restriction and partial order, we are ready to set up the consumption problem and rationalizability problem for this setting. An agent makes a choice from a compact set $B \subseteq X$. Let $\mathcal{B}$ a collection of compact budget sets and $C : \mathcal{B} \to X$ be the choice function that corresponds to the observed choice from budgets. The dataset is a tuple $(\mathcal{B}, C)$; that is, a collection of choices from budget sets. We denote the downward closure of the budget set by

$$B^{\geq} = \{x \in X : \text{ there is } x' \in B \text{ such that } x' \geq x\}$$

and the strict downward closure of the budget set by

$$B^{>} = \{x \in X : \text{ there is } x' \in B \text{ such that } x' > x\}.$$

Finally, we can define rationalizability of the data set with utility over consumption streams. A data said is said to be **consumption stream rationalizable** if there is a monotone utility function $U : X \to \mathbb{R}$ such that $U(C(B)) \geq U(x)$ for every $x \in B$ and every $B \in \mathcal{B}$.

Corollary 2. *A data set is consumption stream rationalizable if and only if for every sequence $x_1 \in C(B_1), \ldots, x_n \in C(B_n)$, $B_1, \ldots, B_n \in \mathcal{B}$,*

$$x_{t+1} \in B_t^{\geq} \text{ for every } t = 1, \ldots, n-1 \text{ implies } x_1 \notin B_n^{>}.$$

4. Concluding remarks

We conclude by making some remarks about the connection of our work to existing results. First, we show why the construction of Nishimura et al. (2017) cannot be applied to our setting once we drop the local-compactness assumption. Second, we show that our work generalizes some of the results from Reny (2015) by allowing the experiment to be infinite. Third, we link our work to the literature on utility representation of incomplete partial orders.

4.1. Continuity and topological assumptions

The result of Nishimura et al. (2017) relies on the Nachbin (1965) extension theorem, which requires (local-)compactness of the space in order to obtain a continuous utility function.[7] We do not assume local-compactness of the space of the alternatives. Therefore, Nachbin (1965) extension theorem cannot be applied directly to our setting. Further we show why compactness is deeply linked with continuity. Let us show the intuition behind the fact that a continuous partial order[8] in the compact Hausdorff space can be represented by continuous utility function. The key idea is to find disjoint increasing and decreasing[9] open supersets for every two sets (F_1, F_2) such that none of the elements in the first set (F_1) is greater than element in the second one (F_2). These two sets play the role of upper and lower contour sets.

The construction of the open sets F_1, F_2 relies on the compactness of the space. In particular, there are two crucial implications of compactness of the Hausdorff space. First, for every two closed disjoint sets there are open disjoint neighborhoods of these sets.[10] Second, a crucial property is that F_1 appears to be a compact set as well. This fact implies that it can be covered by the union of decreasing open neighborhoods of a *finite amount of points*. Moreover, the *intersection* of the increasing neighborhoods of these points covers F_2. Hence, it is essential to have a finite amount of neighborhoods to make sure that the cover of the F_2 is open.[11] At the same time the finiteness of cover-generating open neighborhoods is a consequence of the compactness of the space.

Hence, the proof used in Nishimura et al. (2017) and Nachbin (1965)

[7] To be precise, Nishimura et al. (2017) uses the Levin (1983) theorem. Although the intuition is the same, considering the Levin (1983) theorem would further complicate the explanation without adding extra intuition of what goes wrong with the construction of the continuous utility function without compactness of the space assumption.

[8] A partial order is said to be continuous if all upper and lower contour sets are closed.

[9] Both properties should be defined with respect to binary relation $\succeq$. Increasing means that if $x \in F$, then every x' such that $x' \succeq x$ implies that $x' \in F$. Decreasing means that if $x \in F$, then every x' such that $x \succeq x'$ implies $x' \in F$.

[10] This observation is a consequence of the fact that compact Hausdorff spaces are normal.

[11] Recall that to guarantee the continuous utility representation of the complete and transitive preference relation it is important to ensure that this relation is continuous as well. That is, all strict upper and lower contour sets are open.

relies on local-compactness of the space, while our result does not require this assumption and requires only separability of the space if the partial order is continuous.[12] In addition we need to assume that the space is connected, that is there are only two sets which are closed and open empty set and X itself.

Corollary 3. *Let X be a connected and separable space endowed with a continuous partial order $\geq$. A data set $(\mathcal{B}, C)$ is rationalizable if and only if it satisfies GARP.*

4.2. Size of the data set

Note that we do not make any assumption about the size of the data set, and still obtain a monotone utility representation of the revealed preference ordering. Hence our results is a generalization of Theorem 1 and Proposition 6 in Reny (2015) which provide a utility representation of a consistent revealed preference relation over the real hyperplane for a data set of arbitrary size. We show that a similar result holds for an arbitrary space endowed with separable partial order where data set contains arbitrary budgets. Examples I and II in Reny (2015) show that if data set is rationalizable and at least countable then there are data sets which cannot be rationalized by upper- or lower-semicontinuous utilities. Hence, our utility representation result is tight with respect to the size of the observed data set.

4.3. Representation of partial orders

Our paper is indirectly connected to the literature dealing with the representation of partial orders. The closest result in this direction is the one of Herden and Levin (2012), who relax the assumption of separability of the partial order and still obtain a utility representation.[13] However, we are not only seeking to represent the partial order $\geq$ but find a representation which is also consistent with revealed preference relation. Thus, we need to look at

[12] A partial order is said to be continuous if lower and upper contour sets of $>$ are open set. Formally lower contour set can be defined as $L_>(x)\{y : y > x\}$ and upper contour set can be defined as $U_>(x)\{y : y > x\}$.

[13] Part of our proof could be simplified using the result of Herden and Levin (2012). We prefer to present a full proof to provide a better insight of the properties of transitive closure which make it possible to construct utility representable extension. This decision allows the reader to think about other possible desiderata (e.g., homotheticity or quasilinearity) to which a similar result can be applied.

the transitive closure of the union of these relations, which does not have to be separable. Our starting point is constructing a separable extension of this relation. Hence, our main result can be considered as a further weakening of separability conditions for the partial order to be utility representable.

Appendix A. Proofs

Before we proceed with the proof, let us introduce some additional notation. We first lay out basic definitions related to the preference relations and extensions. Next, we present the nomenclature and supplementary results for the transitive closure as a function over preferences. Finally, we present the formal proof of Theorem 1.

Appendix A.1. Preferences and extensions

A set $R \subseteq X \times X$ is said to be a binary relation. We denote the reverse relation by $R^{-1} = \{(x, y)|(y, x) \in R\}$. We denote the symmetric (indifferent) part of R by $I(R) = R \cap R^{-1}$ and the asymmetric (strict) part by $P(R) = R \setminus I(R)$. We denote the incomparable part by $N(R) = X \times X \setminus (R \cup R^{-1})$. A binary relation that is reflexive, i.e. $(x, x) \in R$ for every $x \in X$, is said to be a preference relation. We denote the set of all preference relations on X by $\mathcal{R}$.

Note that

$$\geq \equiv \{(x, y) \in X^2 : y \leq x\}$$

is a preference relation, with

$$> \equiv \{(x, y) \in X^2 : y < x\}$$

being its strict part.

Definition A.1. *A preference relation R satisfies:*

- ***completeness*** *if $(x, y) \in R \cup R^{-1}$ for all $x, y \in X$ (or equivalently $N(R) = \emptyset$).*
- ***transitivity*** *if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.*
- ***monotonicity*** *if $\geq \subseteq R$ and $> \subseteq P(R)$.*

– Z-***separability*** *if there is a countable* $Z \subseteq X$ *such that* $(x, y) \in P(R)$ *implies that there is* $z \in Z$ *such that* $(x, z), (z, y) \in R$.

As we already mentioned, Theorem 1 is a preference extension theorem. Hence, in order to proceed further we need to formally define a preference extension.

Definition A.2. *A preference relation* R' *is an* ***extension*** *of* R, *denoted* $R \preceq R'$, *if* $R \subseteq R'$ *and* $P(R) \subseteq P(R')$.

We say that R is **consistent** with R' if $P^{-1}(R) \cap R' = \emptyset$. Next we show that preference relation R' is an extension of $R \subseteq R'$ if and only if R is consistent with R'. Consistency is an operationalizable version of extension which will be extensively used further.

Lemma A.1. *Let* $R \subseteq R'$. $R \preceq R'$ *if and only if* $P^{-1}(R) \cap R' = \emptyset$.

Proof. **(⇒)** Assume $P^{-1}(R) \cap R' \neq \emptyset$, then there is $(x, y) \in P^{-1}(R) \cap R'$. That is $(y, x) \in P(R)$ and $(x, y) \in R'$. At the same time $R \preceq R'$ implies that $(y, x) \in P(R')$, that is a contradiction.

(⇐) Assume that $R \subseteq R'$ but $R \not\preceq R'$, that is $P(R) \not\subseteq P(R')$. Hence, there is $(x, y) \in P(R)$ and $(x, y) \notin P(R')$. At the same time $R \subseteq R'$ implies that $(x, y) \in R'$. Therefore, $(y, x) \in R'$, because $(x, y) \in I(R') = R' \setminus P(R')$. Hence, $(y, x) \in P^{-1}(R) \cap R' \neq \emptyset$. □

Appendix A.2. Transitive closure as function over preferences

Let $T : \mathcal{R} \to \mathcal{R}$ be the transitive closure, defined by $(x, y) \in T(R)$ if and only if there is a finite sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R$. The transitive closure is an example of a function over preference relations, which we develop in what follows.

Lemma A.2 (Demuynck (2009))**.** $R = T(R)$ *if and only if* R *is transitive.*

Definition A.3. *For any given function* $F : \mathcal{R} \to \mathcal{R}$, *we let*

– $\mathcal{R}_F = \{R \in \mathcal{R} | R \preceq F(R)\}$,

– $\mathcal{R}_F^Z = \{R \in \mathcal{R}$ *and* R *is* Z*-separable* $| R \preceq F(R)\}$.

$\mathcal{R}_F$ is a set of preferences which are consistent with F, that is every $R \in \mathcal{R}_F$ can be extended by taking $F(R)$. $\mathcal{R}_F^Z$ is a set of consistent with F preference relations which are also Z-separable. Next, we define a set of properties of function over preference relations which guarantee existence of complete fixed point extension of every consistent preference relation which can be represented by a utility function.

Definition A.4. *A function $F : \mathcal{R} \to \mathcal{R}$ is said to be*

- ***monotone*** *if $R \subseteq R'$ implies $F(R) \subseteq F(R')$ for all $R, R' \in \mathcal{R}$,*
- ***closed*** *if $R \subseteq F(R)$ for all $R \in \mathcal{R}$,*
- ***idempotent*** *if $F(F(R)) = F(R)$ for all $R \in \mathcal{R}$,*
- ***algebraic*** *if for all $R \in \mathcal{R}$ and all $(x, y) \in F(R)$, there is a finite relation $R' \subseteq R$ such that $(x, y) \in F(R')$,*
- ***expansive*** *if for every $R = F(R)$ such that $N(R) \neq \emptyset$, there is a nonempty set $S \subseteq N(R)$ such that $R \cup S \in \mathcal{R}_F$ and $P(R) = P(R \cup S)$,*[14]
- ***transitivity-inducing*** *if $T(F(R)) = F(R)$ for every R, equivalently, every $F(R)$ is transitive relation.*
- ***separability-preserving*** *with respect to some countable set Z, if $R \in \mathcal{R}_F^Z$ then $F(R) \in \mathcal{R}_F^Z$.*

The first four properties define an algebraic closure (see Demuynck, 2009). Further we refer to a function which satisfy all of the properties above as a **rational closure**. Results stated below hold for rational closures and sometimes even for wider classes of the functions over preference relations. As we show below, the transitive closure is rational.

Lemma A.3. *T is a rational closure.*

For the proof that T is an algebraic closure as well as for the proof that every fixed point of transitive closure is transitive see Demuynck (2009). Since every fixed point of T is transitive (see Lemma A.2), then T is transitivity-inducing. Hence, it remains to be shown that T is expansive and separability-preserving.

[14]Equivalently one can just say that $S = I(S)$. We use less straight-forward definition since that is the implication we actually need in the proof.

Proof.

T is expansive
Consider a relation $R = T(R)$ and assume that $N(R) \neq \emptyset$. Take any element $(x, y) \in N(R)$ and consider the relation $R' = R \cup \{(x, y), (y, x)\}$. We claim that $R' \preceq T(R')$, which would prove that T is expansive. $R' \subseteq T(R')$ since T is closed. Therefore, we left to show that $P(R') \subseteq P(T(R'))$. Assume, on the contrary, that there are elements z and w for which $(z, w) \in P(R')$ and $(w, z) \in T(R')$. Note that $(x, y) \neq (z, w) \neq (y, x)$ since $(z, w) \in P(R')$ and $(x, y), (y, x) \in I(R')$. From the definition of T, we know that there is some finite sequence $s_1, \ldots, s_n$ such that $s_1 = w$, $s_n = z$, and $(s_j, s_{j+1}) \in R'$ for each $j = 1, \ldots, n-1$. Let m be the minimal (integer) length of such sequence, and let S be any such sequences of length m.

Given a sequence S as described above, there is some j such that either $(s_j, s_{j+1}) = (x, y)$ or $(s_j, s_{j+1}) = (y, x)$ for some $1 < j < m-1$; otherwise $(w, z) \in T(R) = R$, contradicting $(z, w) \in P(R')$. Suppose without loss of generality that $(s_j, s_{j+1}) = (x, y)$ for some $1 < j < m-1$; then there is no $k \neq j$ such that $(s_k, s_{k+1}) = (y, x)$ or $(s_k, s_{k+1}) = (x, y)$, otherwise S would not be a shortest sequence that connects w and z such that every consecutive pair is in R'. Since $(z, w) \in P(R')$, we have $(z, w) \in R'$. Now consider the finite sequence $y, s_{j+2}, \ldots, s_{m-1}, z, w, s_1, \ldots, s_{j-1}, x$. Every pair of consecutive elements of the sequence is in R' and is different from (x, y) and (y, x), so every pair of consecutive elements of the sequence is in R. Then $(y, x) \in T(R) = R$, contradicting $(x, y) \in N(R)$.

Finally, let us note that $P(R') = P(R)$ since we only added the indifferent comparison between x and y to the preference relation, i.e. $R' = R \cup S$, where $S = I(S)$. Given that $R' = I(R') \cup P(R')$ (by construction of the sets), it is immediate that $P(R') = P(R)$.

T is separability-preserving
Take Z such that R is Z-separable. Take $(x, y) \in P(T(R))$. Then there is a sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R$ for all $j = 1, \ldots, n-1$ and an index k such that $(s_k, s_{k+1}) \in P(R)$. Z-separability of R implies that there is $z \in Z$ such that $(s_k, z), (z, s_{k+1}) \in R$. Moreover, by construction, $(x, z), (z, y) \in T(R)$. Hence, $T(R)$ is also separable. □

Lemma A.4. *For every $R, R' \in \mathcal{R}$, $T(T(R) \cup T(R')) = T(R \cup R')$.*

Proof. Since $T(R)$ is closed, then $R \subseteq T(R)$ and $R' \subseteq T(R')$. Since $T(R)$ is monotone, then $T(R \cup R') \subseteq T(T(R) \cup T(R'))$. Hence, we are left to show that $T(R \cup R') \supseteq T(T(R) \cup T(R'))$ to complete the proof.

Take $(x, y) \in T(T(R) \cup T(R'))$. Hence, by construction of transitive closure there is $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in T(R) \cup T(R')$ for every $j \in \{1, \ldots, n-1\}$. Without loss of generality assume that $(s_j, s_{j+1}) \in T(R)$ for some j, then there is a sequence $s_j = s'_1, \ldots, s'_m = s_{j+1}$, such that $(s'_k, s'_{k+1}) \in R$. Hence, for every such j we can merge the sequence to the original one and construct it such that $x = \bar{s}_1, \ldots, \bar{s}_N = y$ such that $(\bar{s}_j, \bar{s}_{j+1}) \in R \cup R'$ for every $j \in \{1, \ldots, N-1\}$. Hence, $(x, y) \in T(R \cup R')$. $\square$

Appendix A.3. Proof of Theorem 1

We start from laying out the schema of the proof. For this purpose we define a revealed preference relation denoted by R_E. Given a data set $E = (\mathcal{B}, C)$, let $(x, y) \in R_E$ if there is $B \in \mathcal{B}$ such that $x \in C(B)$ and $y \in B$. That is, y belongs to some budget from which x is chosen (recall that $\mathcal{B}$ denotes the set of budgets from which the choices are observed).

The proof proceeds as follows:

1. GARP is equivalent to $>^{-1} \cap T(R_E \cup \geq) = \emptyset$.
2. If data is rationalizable then $>^{-1} \cap T(R_E \cup \geq) = \emptyset$.
3. If $>^{-1} \cap T(R_E \cup \geq) = \emptyset$, then data is rationalizable:
   a. If $>^{-1} \cap T(R_E \cup \geq) = \emptyset$, then there is a separable and transitive R such that $> \subseteq P(R)$ and $T(R_E \cup \geq) \subseteq R$.
   b. If there is separable and transitive R such that $> \subseteq P(R)$ and $T(R_E \cup \geq) \subseteq R$, then there is a complete, separable and transitive R^* such that $> \subseteq P(R^*)$ and $T(R_E \cup \geq) \subseteq R^*$.
   c. If there is a complete transitive and separable preference relation R^* such that $> \subseteq P(R^*)$ and $T(R_E \cup \geq) \subseteq R^*$, then data set is rationalizable.[15]

1. GARP is equivalent to $>^{-1} \cap T(R_E \cup \geq) = \emptyset$.

[15] Steps 3b and 3c of the proof can be omitted by referring to the Theorem 3.1 from Herden and Levin (2012).

Lemma A.5. *The data set $E = (\mathcal{B}, C)$ satisfies GARP if and only if* $>^{-1} \cap T(R_E \cup \geq)$.

Before we start the proof let us make a simple observation on the nature of transitive closure of the revealed preference relation. Let $x = s_1, \ldots, s_n = y$ be a shortest sequence such that $(s_j, s_{j+1}) \in R_E \cup \geq$, then there is no $j \in \{1, \ldots, n-1\}$ such that $s_j \geq s_{j+1} \geq s_{j+2}$. Otherwise the sequence can be shortened since the partial order $\geq$ is transitive. This implies that at least one of s_j and s_{j+1} is a chosen point for every $j \in \{1, \ldots, n-1\}$.

Proof. **(⇒)** Assume that there is $(x, y) \in >^{-1} \cap T(R_E \cup \geq)$ so there is a violation of consistency. Consider some shortest sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R_E \cup \geq$ which adds (x, y) to $T(R_E \cup \geq)$. Hence, there is a subsequence $s'_1, \ldots, s'_m = y$ such that $s'_{k+1} \in B_k^{\geq}$ and $s'_k \in C(B_k^{\geq})$ for every $k \in \{1, \ldots, m-1\}$. Since $x = s_1$, then construction of the subsequence (of a shortest sequence) implies that either $s_1 \in C(B_1)$, then $s'_1 = s_1$; or $s_1 \geq s_2$ and $s_2 \in C(B_2)$, then $s_2 = s'_1$. Hence, $x \geq s'_1$. At the same time $y > x \geq s'_1$, that is either $y \in C(B_m)$ and $s'_1 \in B_m^{>}$, or $y \in B_{m-1}^{\geq}$ which implies $s'_1 \in B_{m-1}^{>}$. Either of the cases generates a violation of GARP.

(⇐) Assume there is a sequence $x_1 \in C(B_1), \ldots, x_n \in C(B_n)$ such that $B_1, \ldots, B_n \in \mathcal{B}$, $x_{j+1} \in B_j^{\geq}$ for $j = 1, \ldots, n-1$, and $x_1 \in B_n^{>}$, so there is a violation of GARP. Recall that $x_{j+1} \in B_j^{\geq}$ implies that there is $y_j \in B_j$ such that $y_j \geq x_{j+1}$. By construction of revealed preference relation R_E, $(x_j, y_j) \in R_E$ since $x_j \in C(B_j)$ and $y_j \in B_j$ for every $j \in \{1, \ldots, n-1\}$. Moreover, $x_1 \in B_n^{>}$ implies that there is $y_n \in B_n$ such that $y_n > x_1$. Hence, we can construct a sequence $x_1 = s_1, \ldots, s_m = y_n$ such that $x_j = s_j$ and $s_{j+1} = y_{j+1}$ for odd $j \in \{1, \ldots, m-1\}$. By construction of this sequence $(s_j, s_{j+1}) \in R_E \cup \geq$ for every $j \in \{1, \ldots, m-1\}$. Hence, $(x_1, y_n) \in T(R_E \cup \geq)$. At the same time, we know that $(y_n, x_1) \in >$. That is a direct contradiction of consistency since $(x_1, y_n) \in >^{-1} \cap T(R_E \cup \geq)$. □

2. If the data is rationalizable then $>^{-1} \cap T(R_E \cup \geq) = \emptyset$.

Lemma A.6. *If the data is rationalizable then* $>^{-1} \cap T(R_E \cup \geq) = \emptyset$.

Proof. We proceed by contradiction. Suppose there is $(y, x) \in >$ and $(x, y) \in T(R_E \cup \geq)$. Consider some shortest sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R_E \cup \geq$ for $j \in \{1, \ldots, n-1\}$. If the choices are generated by a monotonic utility function, then $(s_j, s_{j+1}) \in R_E \cup \geq$ implies that $u(s_j) \geq u(s_{j+1})$. Hence, we can conclude that $u(x) \geq u(y)$, by transitivity of $\geq$. At the same time $y > x$, hence monotonicity implies that $u(y) > u(x)$, that is a contradiction. □

3. If $>^{-1} \cap T(R_E \cup \geq) = \emptyset$ then data is rationalizable.

Before we proceed with the proof let us state an auxiliary lemma which we use further.

Lemma A.7. *For every chain*

$$R_0 \preceq R_1 \preceq \cdots \preceq R_\alpha \preceq \cdots$$

such that $R_\alpha \in \mathcal{R}_T$ for all α, we have $\overline{R} = \bigcup_{\alpha \geq 0} R_\alpha \in \mathcal{R}_T$.[16]

Proof. On the contrary assume that there is $(x, y) \in T(\overline{R})$ but $(y, x) \in P(\overline{R})$. By construction of $\overline{R}$ we know that $(y, x) \in R_a$ for some relation R_a, and therefore $(y, x) \in R_\alpha$ for $\alpha \geq a$. Since T is algebraic, there is some finite relation $R' \subseteq \overline{R}$ such that $(x, y) \in T(R')$. Moreover, since R' is finite, there is some R_b in the chain such that $R' \subseteq R_b$. Since T is monotone, $T(R') \subseteq T(R_b)$ and therefore $(x, y) \in T(R_b)$. Monotonicity implies $(x, y) \in T(R_\alpha) \supseteq T(R_b)$ for $\alpha \geq b$. Hence, there is a $c \geq \max\{a, b\}$ such that R_c is not consistent, that is a contradiction.[17] □

3a. If $>^{-1} \cap T(R_E \cup \geq) = \emptyset$ and $\geq$ is separable, then there is a separable and transitive $\hat{R}$ such that $> \subseteq P(\hat{R})$ and $T(R_E \cup \geq) \subseteq \hat{R}$.

Before we proceed with the proof let us introduce some more of additional notation. Let $R = T(R)$ be a transitive relation such that $> \subseteq P(R)$ and $T(R_E \cup \geq) \subseteq R$. Denote by

$$\begin{aligned} SNS(R) = \{(x, y) :\ & (y, x) \in P(R) \text{ for every sequence } y = s_1, \ldots, s_n = x \\ & \text{such that } (s_j, s_{j+1}) \in R \text{ there is no } k \in \{1, \ldots, n-1\} \\ & \text{such that } (s_k, s_{k+1}) \in >\} \end{aligned}$$

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[16]Note that here and further we talk about possibly uncountable sequences of preference relations, since the sequence could add the entire (uncountable) sequence of comparisons, one-by-one.

[17]The same proof can be conducted for any closed, monotone and algebraic function $F : \mathcal{R} \to \mathcal{R}$.

the set of strict but not separated pairs.

Note that it is only defined for the fixed points of $T(R) = R$ that extend the separable partial order $\geq$. Since $\geq$ is a separable partial order, for every $(x, y) \in >$, there is $z \in Z$ (recall that Z is a countable set) such that $(x, z), (z, y) \in \geq$. Since we want to obtain separable extension of $T(R)$, we need to define the set of pairs which cannot be separated. Since we only consider fixed points of T we know that for every $(x, y) \in T(R)$ there is a sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in R \succeq \geq$. Hence, if there is $k \in \{1, \ldots, n-1\}$ such that $(s_k, s_{k+1}) \in >$, then there is $z \in Z$ such that $(s_k, z), (z, s_{k+1}) \in \geq$. Then, by construction of the transitive closure $(x, z), (z, y) \in T(R) = R$, i.e. the pair $(x, y) \in P(R)$ can be separated by $z \in Z$. Hence, the non-separated points are such for which there is no $(s_k, s_{k+1}) \in >$ for every sequence which adds (x, y) to R.

Before we proceed with the proof, note that $SNS(T(R_E \cup \geq))$ is a nontrivial object. Recall that $(x, y) \in R_E$ is $x \in C(B)$ and $y \in B$ for some $B \in \mathcal{B}$. Hence, technically there can be an $(x, y) \in P(R_E)$ such that $(x, y) \in P(T(R_E \cup \geq))$ and there is no $z \in Z$ to separate x and y. Therefore, one way to show that there is separable $\hat{R} \supseteq T(R_E \cup \geq)$ and $\geq \preceq \hat{R}$ is to make sure that $SNS(T(\hat{R})) = \emptyset$.[18]

Denote

$$\bar{\Omega} = \{R = T(R) : \ T(R_E \cup \geq) \subseteq R \text{ and } > \subseteq P(R)\}.$$

If $>^{-1} \cap T(R_E \cup \geq) = \emptyset$ then $\bar{\Omega}$ is nonempty. This observation follows from the fact that $T(R_E \cup \geq) = T(T(R_E \cup \geq))$ due to idempotence of $T(R)$ and the consistency condition implies that $> \subseteq P(R_E \cup \geq)$. Next we state some auxiliary results to obtain the goal of this part of the proof.

Lemma A.8. *For every*

$$R_0 \subseteq R_1 \subseteq \ldots \subseteq R_\alpha \subseteq \ldots$$

such that $R_\alpha \in \bar{\Omega}$ *for all* $\alpha \geq 0$ *we have* $\bar{R} = \bigcup_{\alpha \geq 0} R_\alpha \in \bar{\Omega}$.

[18]This is not the only way to obtain separability. For instance, if there are only countable non-separated points, it suffices to add them to the relation. However, the way we propose is more general and independent of the cardinality of $SNS(R)$.

Proof. From Lemma A.7 we already know that $\bar{R} \in \mathcal{R}_T$ that is $\bar{R} \preceq T(\bar{R})$. Hence we are left to show two properties: (i) $T(\bar{R}) \subseteq \bar{R}$, and (ii) $> \subseteq P(R)$.[19]

$\mathbf{T(\bar{R}) \subseteq \bar{R}}$

Suppose on the contrary $(x, y) \in T(\bar{R})$ and $(x, y) \notin \bar{R}$. Recall that $T(R)$ is algebraic and therefore, there is a finite $R' \subseteq \bar{R}$ such that $(x, y) \in T(R')$. Since R' is finite, then there is $a \geq 0$ such that $R' \subseteq R_a$. At the same time $R_a \in \bar{\Omega}$, hence, $R_a = T(R_a)$ and therefore, $(x, y) \in T(R_a) = R_a \subseteq \bar{R}$.

$> \subseteq \mathbf{P(\bar{R})}$

Assume on the contrary that $(z, w) \in >^{-1} \cap \bar{R} \neq \emptyset$. Given the construction of $\bar{R}$, there is $a \geq 0$ such that $(z, w) \in R_a$, hence, $>^{-1} \cap R_a \neq \emptyset$. That is a contradiction to the fact that $R_a \in \bar{\Omega}$ and hence $> \subseteq P(\bar{R})$.

□

Lemma A.9. *Let $R \in \bar{\Omega}$ and $SNS(R) \neq \emptyset$, then exists $R' \in \bar{\Omega}$ such that $R \subset R'$ and $> \subset P(R')$.*

Proof. Take $(x, y) \in SNS(R)$ and let $R' = T(R \cup \{(x, y)\})$. Since $T : \mathcal{R} \to \mathcal{R}$ is closed and monotone we know that $T(R) = R \subset R' = T(R \cup \{(x, y)\})$. Next we show that $> \subseteq P(R')$. On the contrary assume that $(z, w) \in >^{-1} \cap T(R \cup \{(x, y)\})$. At the same time $(z, w) \notin T(R)$, since $R = T(R)$ and $> \subseteq P(R)$. Hence, there is a shortest sequence $z = s_1, \ldots, s_n = w$ such that $(s_j, s_{j+1}) \in R \cup \{(x, y)\}$ for every $j \in \{1, \ldots, n-1\}$ and there is $k \leq n-1$ and $(s_k, s_{k+1}) = (x, y)$.[20] Hence, we can reorder the sequence, such that

$$y = s'_1, \ldots, w, z, \ldots, s'_n = x$$

and $(s'_j, s'_{j+1}) \in R$ for every $j \in \{1, \ldots, n-1\}$. Since $(x, y) \in SNS(R)$, every sequence that adds (y, x) to $T(R)$ should not contain $(s_k, s_{k+1}) \in >$. At the same time $(w, z) \in >$ is a contradiction to the fact that $(x, y) \in SNS(R)$. Therefore, $> \subseteq P(R')$ and hence, $R' \in \bar{\Omega}$. □

Lemma A.10. *If $>^{-1} \cap T(R_E \cup \geq) = \emptyset$ and $\geq$ is separable, then there is a separable and transitive $\hat{R}$ such that $> \subseteq P(\hat{R})$ and $T(R_E \cup \geq) \subseteq \hat{R}$.*

[19] Although we consider chains ordered with $\preceq$ in Lemma A.7, we could use the same argument using $\subseteq$-ordered chains (see Demuynck, 2009).

[20] Same argument as in Lemma A.2 applies to show that (x, y) enters a shortest sequence only once.

Proof. Since $(\bar{\Omega}, \subseteq)$ is a partially ordered set and from Lemma A.8 we know that every chain has a maximal element within $\bar{\Omega}$. Hence, applying Zorn's Lemma we know that there is a maximal element $\hat{R} \in \bar{\Omega}$. Since $\hat{R} \in \bar{\Omega}$, then it is a transitive (since all $R \in \bar{\Omega}$ are fixed points of T), $> \preceq \hat{R}$ and $T(R_E \cup \geq) \subseteq \hat{R}$. Hence, we are left to show that $\hat{R}$ is separable.

If $SNS(\hat{R}) \neq \emptyset$, then we get immediate contradiction to the fact that $\hat{R}$ is a maximal element given Lemma A.9. Further we consider $SNS(\hat{R}) = \emptyset$. Take $(x, y) \in P(\hat{R})$, given that $\hat{R} = T(\hat{R})$ there is a sequence $x = s_1, \ldots, s_n = y$ such that $(s_j, s_{j+1}) \in \hat{R}$ for every $j \in \{1, \ldots, n-1\}$. Moreover $SNS(\hat{R}) = \emptyset$ implies that there is $k \leq n-1$ such that $(s_k, s_{k+1}) \in >$. Since $\geq$ is separable, then there is $z \in Z$ such that $(s_k, z), (z, s_{k+1}) \in \geq \subseteq \hat{R}$. Given that $\hat{R} = T(\hat{R})$ is a transitive relation, $(x, z), (z, y) \in \hat{R}$. Hence, $\hat{R}$ is separable.

□

3b. If there is separable and transitive R such that $> \subseteq P(R)$ and $T(R_E \cup \geq) \subseteq R$, then there is a complete, separable and transitive R^* such that $> \subseteq P(R^*)$ and $T(R_E \cup \geq) \subseteq R^*$.

Lemma A.11. *For any countable Z and every chain*

$$R_0 \preceq R_1 \preceq \cdots \preceq R_\alpha \preceq \cdots$$

such that $R_\alpha \in \mathcal{R}_T^Z$ for all α, we have $\overline{R} = \bigcup_{\alpha \geq 0} R_\alpha \in \mathcal{R}_T^Z$.

Proof. We know that each element R_α of the chain is consistent with $T(R_\alpha)$ and Z-separable. Hence let us to show that $\overline{R}$ is consistent and Z-separable. Lemma A.7 already shows that $\overline{R}$ is consistent. Hence, we are only left to show that $\bar{R}$ is separable.[21]

$\overline{R}$ is Z-separable

Take $(x, y) \in P(\overline{R})$. By construction of $\overline{R}$ we know that $(x, y) \in R_d$ for some relation R_d, and $(y, x) \notin R_\alpha$ for any α. Hence $(x, y) \in P(R_d)$. Since R_d is Z-separable, there is $z \in Z$ such that $(x, z) \in R_d$ and $(z, y) \in R_d$. Then $(x, z) \in \overline{R}$ and $(z, y) \in \overline{R}$. □

[21] The same proof can be conducted for any closed, monotone and algebraic function $F : \mathcal{R} \to \mathcal{R}$.

Lemma A.12. *If there is separable and transitive $\hat{R}$ such that $>\subseteq P(\hat{R})$ and $T(R_E\cup \geq) \subseteq \hat{R}$, then there is a complete, separable and transitive R^* such that $>\subseteq P(R^*)$ and $T(R_E\cup \geq) \subseteq R^*$.*

Proof. Let

$$\Omega(\hat{R}) = \{R' \in \mathcal{R}_T^Z : \hat{R} \preceq R'\}$$

be the set of extensions of $\hat{R}$ that are themselves Z-separable and can be extended by T. Since $\hat{R}$ is transitive, it is a fixed point of T. That is $\hat{R} = T(\hat{R})$, every preference relation that extends $\hat{R}$ also extends $T(\hat{R})$. Clearly, $\preceq$ is a partial order (reflexive, antisymmetric and transitive binary relation) on $\Omega(\hat{R})$ and we just showed that every chain has an upper bound. Hence, Zorn's lemma, implies that there is a maximal element of $\Omega(\hat{R})$, which we denote by R^*. Since $R^* \in \Omega(\hat{R})$ it is separable. Hence, we are left to show that R^* is complete and transitive.[22]

R^* is complete
To see this, assume on the contrary that $N(R^*) \neq \emptyset$. If R^* is a fixed-point of T, then since T is expansive there is $S \subseteq N(R^*)$ such that $R^* \cup S \in \mathcal{R}_T$ and $P(R) = P(R \cup S)$. Latter fact also guarantees that $R^* \cup S$ is Z-separable, hence $R^* \cup S \in \mathcal{R}_T^Z$ and $R^* \preceq R^* \cup S$, that is a contradiction to R^* being a maximal element of $\Omega(\hat{R})$. If R^* is not a fixed point of T, then $T(R^*)$ is an extension of R^* which is Z-separable (since T is separability-preserving). That is also a contradiction to the fact that R^* is a maximal element of $\Omega(\hat{R})$.

R^* is transitive
In order to show that R^* is transitive we show that it is a fixed point of T and therefore is transitive (see Lemma A.3). $R^* \subseteq T(R^*)$ since $R^* \preceq T(R^*)$. Assume on the contrary that $R^* \subset T(R^*)$. That is there is $(x, y) \in T(R^*)$ and $(x, y) \notin R^*$. At the same time completeness of R^* implies $(y, x) \in P(R^*)$. Hence, $(x, y) \in P^{-1}(R^*) \cap T(R^*) \neq \emptyset$ that contradicts to the fact that $R^* \preceq T(R^*)$. Therefore, $R^* = T(R^*)$. □

3c. If there is complete, transitive and separable R^* such that $>\subseteq P(R^*)$ and $T(R_E\cup \geq) \subseteq R^*$, then the data set is rationalizable.

[22] The same proof can be conducted for an arbitrary rational closure $F : \mathcal{R} \to \mathcal{R}$.

Lemma A.13 (Lemma II in Debreu (1954))**.** *R is a complete transitive and separable relation, then there is a utility function $u : X \to \mathbb{R}$ such that $u(x) \geq u(y)$ if and only if $(x, y) \in R$.*

Lemma A.14. *If there is complete, transitive and separable R^* such that $> \subseteq P(R^*)$ and $T(R_E \cup \geq) \subseteq R^*$, then the data set $E = (\mathcal{B}, C)$ is rationalizable.*

Proof. Suppose there is such R^* as stated in the lemma. The existence of a utility function that represents R^* is immediately guaranteed by Lemma A.13. Since $> \subseteq P(R^*)$ and $\geq \subseteq R^*$, the utility function is monotonic. Finally, $R_E \subseteq R^*$ implies that $(x, y) \in R^*$ for every $x \in C(B)$ and $y \in B$. This in turn implies that $u(x) \geq u(y)$ for every $x \in C(B)$ and $y \in B$. □

Appendix A.4. Separability of Partial Order for the Spaces of Functions

We consider the space of step-functions, that is a functions which can be written as a finite linear combination of indicator functions over intervals. Next, we define it more formally. Let $X \subseteq \mathbb{N}$ be domain for the step function. Let

$$\xi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

be an indicator function, where $A \subseteq \mathbb{N}$ is an interval. Given the domain considered we allow the interval to include a single point. Moreover, for simplicity we consider the intervals to be left-closed and right-open, i.e. having a shape of $[\underline{a}, \bar{a}[$. The same argument can be made without this assumption but would require significant abuse of notation. Hence, a step function can be defined as

$$f(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x)$$

where $n \geq 0$, $\alpha_i \in \mathbb{R}$, and A_i are intervals such that they are

1. pairwise disjoint: $A_i \cap A_j = \emptyset$ for $i \neq j$,
2. exhaust the domain: $\bigcup_{i=1}^{n} A_i = X$.

Denote by F the space of step functions. We define a partial order over the set of step functions as

$$\bar{f} \geq \underline{f} \text{ if } \bar{f}(x) \geq \underline{f}(x) \text{ for every } x \in X,$$

and let

$$\bar{f} > \underline{f} \text{ if } \bar{f}(x) > \underline{f}(x) \text{ for every } x \in X.$$

Next we construct a countable set Z of functions in order to show that the partial order $\geq$ is separable. In order to do so we restrict the set of values to be rational, that is

$$f(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x),$$

where $\alpha_i \in \mathbb{Q}$ for any finite n and finite collection of intervals. Hence, the set Z is countable, since it can be obtained as set of all finite subsets of a countable set.

Lemma A.15. *A partial order $\geq$ over F is separable.*

Proof. Consider $\bar{f} > \underline{f}$ and refer to $\bar{n}$ and $\underline{n}$ the number of intervals corresponding to the $\bar{f}$ and $\underline{f}$ and refer to $\bar{A}$ and $\underline{A}$ the intervals corresponding to the $\bar{f}$ and $\underline{f}$ correspondingly. Recall that every interval can be specified with its borders, hence, the collection of intervals can be characterized by the vector of ordered points from 0 to ∞. We refer to the borders of these intervals as $\bar{a}_i$ and $\underline{a}_i$ (using the lower bound of the interval) Let us construct the vector

$$(a_1, a_2, \ldots, a_n, \infty) \text{ such that} a_i \in \bigcup_{i \in \underline{n}} \{\underline{a}_i\} \cup \bigcup_{i \in \bar{n}} \{\bar{a}_i\}$$

Hence, an interval can be defined as

$$A_i = [a_i, a_{i+1}[\text{ where } a_{n+1} = \infty,$$

where $\alpha_i \in [\underline{f}(x), \bar{f}(x)] \cap \mathbb{Q}$ for any $x \in A_i$.

To finalize the proof that f separates $\underline{f}$ and $\bar{f}$ let us make an important observation. The collection of intervals A_i is finer than both $\underline{A}_i$ and $\bar{A}_i$ by construction. That is, for every A_i there are $\underline{A}_j$ and $\bar{A}_k$ such that $A_i \subseteq \underline{A}_j$ and $A_i \subseteq \bar{A}_k$. This fact implies that both $\underline{f}(x) = \underline{f}(x')$ and $\bar{f}(x) = \bar{f}(x')$ for every $x, x' \in A_i$. Hence, by construction

$$\bar{f}(x) \geq f(x) \geq \underline{f}(x) \text{ for every } x \in A_i \text{ for every } i \leq n,$$

and therefore, for every $x \in X$ since $\bigcup_{i=1}^{n} A_i = X$. □

Note that Lemma A.15 immediately implies the proofs for Corollaries 1 and 2. Every cumulative distribution function defined by a mixed strategy is already a set function. For the separability of the partial order ($\geq_{FSD}$) we need to make a little adjustment. In the language of step-functions, the strict part of the partial order defined as follows.

$$F_\sigma >_{FSD} F_{\sigma'} \text{ if } \begin{cases} F_\sigma(x) = F_{\sigma'}(x) & \text{for } F_\sigma(x) = F_\sigma(x) = 1 \\ F_\sigma(x) = F_{\sigma'}(x) & \text{for } F_\sigma(x) = F_\sigma(x) = 0 \\ F_\sigma(x) < F_{\sigma'}(x) & \text{for } \textit{otherwise} \end{cases}$$

Hence, we can use the same argument as in Lemma A.15 since 0 and 1 are in $\mathbb{Q}$ and therefore, we would not need to expand the set Z. That is, whenever we construct the f such that $F_\sigma \geq_{FSD} f \geq_{FSD} F_{\sigma'}$ we just set

$$f(x) = 1 \text{ if } F_\sigma(x) = F_\sigma(x) = 1 \text{ and } f(x) = 0 \text{ if } F_\sigma(x) = F_\sigma(x) = 0,$$

and the rest of the construction goes the same way as before.

Every consumption stream is clearly a step function. For every period until t_r, the interval is defined as $A_t = [x_t, x_{t+1}[$. For the periods from t_r and until ∞ the interval is defined as $A_{t_r} = [x_{t_r}, \infty[$. Hence, we can define a consumption stream as a step function as

$$f_x = \sum_{t=1}^{t_r} x_t \chi_{A_t}.$$

Finally, the partial order $\geq$ over the space of consumption streams implies the corresponding partial order over the step functions, that is

$$x_t > x'_t \text{ implies } f_x(t) > f_{x'}(t) \text{ for every } t \in T.$$

Appendix A.5. Proof of Corollary 3

Proof. Since the space X is separable, there is a countable set $Z \subseteq X$ such that every open set in X contains at least one element of Z. We are going to use Z as the countable set for the partial order $\geq$ as well. That is, we need to show that for every $x > x'$ there is $z \in Z$ such that $x \geq z \geq x'$. Recall that $L_>(x)$ is the lower contour set of x and $U_>(x')$ is the upper contour set of x'. Hence, we start from showing that

$$T = L_>(x) \cap U_>(x')$$

is a non-empty set.

Assume on the contrary that T is an empty set. Then, $U_{\geq}(x) = U_{>}(x') \neq \emptyset$. At the same time $U_{>}(x')$ is an open set, since $\geq$ is a continuous order and $U_{\geq} = X \setminus L_{>}(x)$ is a closed set. This set is clopen (closed and open) and it is clearly neither $\emptyset$ (contains x) nor X itself (x' is excluded). That is a contradiction to the fact that X is a connected space, since connected space contains only two clopen sets, which are $\emptyset$ and X itself.

Hence, $T \neq \emptyset$ and it is an open set as it is an intersection of two open sets. Since X is a separable space, then there is $z \in T \cap Z$. Then, $z \in L_{>}(x)$ implies that $x \geq z$ and $z \in U_{>}(x')$ implies that $z \geq x'$, i.e. we obtain

$$x \geq z \geq x'.$$

□

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